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Controller design for unmanned surface vessels through pure prolongations

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Resumen

Este artículo estudia la planitud diferencial del modelo de seis grados de libertad (6-DOF) de un Barco de Superficie No Tripulado (USV, por sus siglas en inglés), para el cual no se había establecido previamente un resultado definitivo debido a los complejos acoplamientos hidrodinámicos. Demostramos que el modelo general del USV es diferencialmente plano mediante su descomposición en un subsistema de cuatro grados de libertad, denominado sistema núcleo ("core system"), que es plano por pura prolongación, junto con una extensión dinámica endógena.

Este resultado se obtiene a partir de una aplicación sistemática de un algoritmo desarrollado recientemente e implementado por los autores en Matlab. El modelo completo hereda las salidas planas del sistema núcleo independientemente de los valores específicos de los parámetros hidrodinámicos.

Palabras clave: Sistemas de Control no Lineales, Vehículos y Sistemas Marinos Autónomos, Control Geométrico y Estructural, Aplicaciones del Análisis y Diseño no Lineal, Generación y Seguimiento de Trayectorias para Vehículos Autónomos.

Diseño de controladores para vehículos marítimos de superficie autónomos mediante prolongaciones puras

Abstract

This paper studies the differential flatness of the six-degree-of-freedom (6-DOF) model of an Unmanned Surface Vessel (USV), for which no definitive result has previously been established due to complex hydrodynamic couplings. We prove that the general USV model is differentially flat by decomposing it into a four-degree-of-freedom subsystem, referred to as the core system, which is flat by pure prolongation, together with an endogenous dynamic extension.

This result follows from a systematic application of a recently developed algorithm implemented by the authors in Matlab. The full model inherits the flat outputs of the core system independently of the specific values of the hydrodynamic parameters.

Keywords: Non-Linear Control Systems, Autonomous marine systems and vehicles, Structural and geometric control, Application of nonlinear analysis and design, Trajectory and path planning for AVs.

1. Introduction

In recent years, Unmanned Surface Vessels (USVs) have garnered significant attention in the field of marine robotics due to their growing range of applications, including environmental monitoring, surveillance, and autonomous transportation. The effective deployment of these vessels relies heavily on the ability to design precise and robust motion planning and control algorithms. However, USVs pose a challenging control problem, as they are typically underactuated mechanical systems characterized by strong nonlinearities and complex hydrodynamic couplings between their six degrees of freedom (6-DOF).

To address these challenges, the property of differential flatness, introduced by Fliess, Lévine, Martin, and Rouchon in (Fliess et al., 1995, 1999), has proven to be a powerful tool for

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nonlinear systems. A system is considered differentially flat if its states and inputs can be expressed algebraically in terms of a set of specific outputs (flat outputs) and their derivatives. This property significantly simplifies trajectory planning: instead of solving complex differential equations, one can plan trajectories in the flat output space and map them back to the control inputs, ensuring the system naturally satisfies its dynamic constraints.

While the differential flatness of simplified USV models (typically restricted to 3-DOF or 4-DOF) is well-documented in the literature, the general 6-DOF model—which accounts for full hydrodynamic damping and added mass effects—is often treated as a more intractable problem, see *e.g.* (de Doná et al., 2018; Degorre et al., 2025; Sira-Ramirez and Agrawal, 2004). In many cases, the general model is assumed to require complex numerical approximations or is categorized as non-flat due to the intricate coupling between sway, surge, and yaw dynamics.

This paper bridges the gap between simplified models and the full physical model of the vessel. We demonstrate that the general 6-DOF USV model is, in fact, differentially flat. Our approach utilizes a systematic algorithm based on pure prolongation to establish the flatness of a reduced 4-DOF model, that we call the *core system*. We then show that the general model can be viewed as an endogenous dynamic extension of this reduced model. Consequently, the flat outputs identified for the *core system* remain valid for the general system, allowing for exact feedforward linearization and efficient trajectory tracking without loss of generality.

1.1. Model of the unmanned surface vessel

The most general model of an unmanned surface vessel, as presented in several works, see for instance (de Doná et al., 2018), (Degorre et al., 2025) and the references therein is given by:

$$\begin{cases} \dot{x} = u\cos\psi - v\sin\psi, \\ \dot{y} = u\sin\psi + v\cos\psi, \\ \dot{\psi} = r, \\ \dot{u} = F_u + avr - \beta_u u, \\ \dot{v} = bur - \beta_v v, \\ \dot{r} = \Gamma_r + cuv - \gamma r. \end{cases} \tag{1}$$

where x, y are the coordinates of the vehicle, $u, v$are the translation speed components, ψ is the heading angle and r is the rotation speed. Regarding the inputs, F_u is the propulsion force while Γ_r is the rotating moment normalized in mass.

A physical interpretation of the parameters is given in the Handbook by (Fossen, 2021). Note that $b = -1/a$.

The *core system* is a simplified four-degree-of-freedom model, presented in (de Doná et al., 2018), obtained from system 1 by removing the fourth and sixth equations and by considering the variables r, u as the system inputs:

$$\begin{cases} \dot{x} = u\cos\psi - v\sin\psi, \\ \dot{y} = u\sin\psi + v\cos\psi, \\ \dot{\psi} = r, \\ \dot{v} = bur - \beta_v v, \end{cases} \tag{2}$$

1.2. Structure of the paper

In the following sections, we will study the differential flatness of the two models, as well as the flatness achieved by pure prolongation.

The paper is structured as follows: Section 2 recalls the concepts of flatness and flatness by pure prolongation. Section 3 outlines the pure prolongation algorithm. The algorithm is then applied to the *core system* (2) in order to get its flat outputs in Section 4. In Section 5, we prove the flatness of the general model using these outputs. The most general case without physical constraints is also considered in this Section. Finally, Section 6 summarizes the main results and proposes directions for future research.

2. Background on Differential Flatness and Flatness by Pure Prolongation

A control system in affine form over a smooth n-dimensional manifold X is given by

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \tag{3}$$

where $x \in X$ is the n-dimensional state vector, with $m < n$ (otherwise, the problem of feedback linearization is trivial) and f and g_i are smooth vector fields in the tangent bundle TX of X for each $u = (u_1, \ldots, u_m)$.

The system is static feedback linearizable if there exists a diffeomorphism $z = \phi(x)$ and a regular feedback law $u = a(x) + b(x)v$ such that, in the new variables, the system becomes linear. A necessary and sufficient condition for a control system to be linearizable by static feedback was given in (Hunt et al., 1983) and (Jakubczyk and Respondek, 1980):

Theorem 1. *System* (3) *is static feedback linearizable if,and only if, The distributions*

$$D_0 = \{g_1, \ldots, g_m\} \quad D_i = \{g_1, \ldots, g_m, \ldots, ad_f^i g_1, \ldots, ad_f^i g_m\}$$

where we use the notation $\{g_1, \ldots, g_m\}$ *for the distribution generated by the vector fields* $g_1, g_2 \ldots, g_m$ *, must be of constant rank and involutive. Moreover, there exists i such that the rank of* D_i *equals n.*[1]

A dynamic system

$$\dot{x} = f(x, u) \tag{4}$$

with m inputs is said to be differentially flat[2] if and only if there exist m functions $y = (y_1, \ldots, y_m)$ (called flat outputs) such that

$$y = Y(x, u, \dot{u}, \ldots) \quad x = X(y, \dot{y}, \ldots) \quad u = U(y, \dot{y}, \ldots) \tag{5}$$

[1] We have used the classical notations $ad_f^0 g = g$ and $ad_f^k g = [f, ad_f^{k-1} g]$ for all $k \geq 1$.

[2] Differential flatness and endogenous dynamic feedback are here defined in an informal way, but otherwise sufficient for our context. The reader may find a rigorous and thorough presentation of these concepts in (Fliess et al., 1999) and(Lévine, 2009) via the notion of Lie-Bäcklund isomorphism.

where the dependence is up to a finite number of derivatives.

Differential flatness was introduced in (Fliess et al., 1995). It has been proven that a system is differentially flat if, and only if, it is dynamic feedback linearizable via an *endogenous dynamic feedback*; see (Martin, 1992; Fliess et al., 1995, 1999; Lévine, 2009) for a detailed description.

Roughly speaking, a dynamic feedback is a feedback of the form

$$u = \alpha(x, z, v), \quad \dot{z} = \beta(x, z, v) \tag{6}$$

with $z = (z_1, \ldots, z_q)$.

It is called endogenous if, and only if, z and v can be expressed as functions $z = Z(x, u, \dot{u}, \ldots)$, $v = V(x, u, \dot{u}, \ldots)$ where, again, the dependence is up to a finite number of derivatives. It is proven in (Martin, 1992; Fliess et al., 1999; Lévine, 2009) that (4) is differentially flat if, and only if, the closed-loop system (4)-(6) is diffeomorphic to the linear system in Brunovský form:

$$y_i^{(r_i)} = v_i, \quad i = 1, \ldots, m \tag{7}$$

for suitably chosen integers r_i, called the controllability indices and where y_i, $i = 1, \ldots, m$ are flat outputs. Indeed, we must have $n + q = \sum_{i=1}^{m} r_i$.

The following elementary sufficient condition for differential flatness will be extensively used in the sequel:

Proposition 1. *Consider the system*

$$\dot{x} = f(x, z) \tag{8}$$

$$\dot{z} = g(x, z, u) \tag{9}$$

where (8) *is differentially flat and* (9) *is an endogenous dynamic feedback, z and u having the same dimension, equal to m. Then, the system* (8)-(9) *is differentially flat with the same flat outputs y as* (8).

Moreover, for (9) *to be endogenous, it suffices that* $\operatorname{rank}\dfrac{\partial g}{\partial u} = m$ *in a dense open set.*

Let y be a flat output for subsystem (8). Thus, $x = X(y, \dot{y}, \ldots)$, $z = Z(y, \dot{y}, \ldots)$, with dependencies up to a finite number of derivatives, and since the subsystem (9), considered as a dynamic feedback, is endogenous, we have, by definition $u = U(x, z, \dot{z}, \ldots)$. Hence, by composition, u is a function of y and successive derivatives up to a finite order, which proves the differential flatness of (8)-(9) with the same flat outputs y as subsystem (8). The last part of the proposition straightforwardly results from the implicit function theorem.

Although necessary and sufficient conditions for verifying differential flatness were outlined in (Lévine, 2011), they do not yield a finite set of criteria for determining whether a given system is differentially flat.

Consider the system (3) with $\dim x = n$, and $\dim u = m$. A pure prolongation of the system is given by the associated prolonged vector field

$$g^{(\mathbf{j})} \doteq g\frac{\partial}{\partial x} + \sum_{i=1}^{m}\sum_{p=0}^{j_i-1} u_i^{(p+1)}\frac{\partial}{\partial u_i^{(p)}},$$

where the prolonged states are $u_i^{(p)}$, $p = 0, \ldots, j_i - 1$, $i = 1, \ldots, m$, and the control inputs $u_i^{(j_i)}$. In other words, the original states remain the same, while the new states added to the system are inputs and derivatives of these inputs up to a certain order. Throughout the paper, the notation $g\frac{\partial}{\partial x}$ will denote the vector field $\sum_{j=1}^{n} g_j \frac{\partial}{\partial x_j}$.

Definition 1. *System (3) is* flat by pure prolongation or linearizable by pure prolongation *(in short P^2-flat) at a point $x_0 \in X \times \mathbb{R}^m_\infty$ if, and only if, there exist finite $\mathbf{j}$ such that the prolonged system of order $\mathbf{j}$ is equivalent by diffeomorphism and feedback to a system in Brunovský form.*

Remark 1. *As already proven in (Sluis and Tilbury, 1996; Franch, 1999; Lévine, 2024), if a system is P^2-flat by means of a prolongation of order $j = (j_1, \ldots, j_m)$, with $j_1 \leq j_2 \leq \cdots \leq j_m$, then it is also P^2-flat with prolongation $0 \leq j_2 - j_1 \leq \cdots \leq j_m - j_1$. Hence, we may assume that, up to input permutation, u_1 is not prolonged.*

3. Pure Prolongation Algorithm

Given a control system of the form

$$\dot{x} = f(x, u)$$

and a multi-integer $\mathbf{j} \triangleq (j_1, \ldots, j_m) \in \mathbb{N}^m$, the prolonged system is defined in the extended space

$$X^{(\mathbf{j})} \triangleq X \times \mathbb{R}^{m+|\mathbf{j}|}$$

with $|\mathbf{j}| = j_1 + \ldots + j_m$ and the new state coordinates are given by $\bar{x}^{(\mathbf{j})} \triangleq (x, \bar{u}^{(\mathbf{j})}) \triangleq$

$$(x_1, \ldots, x_n, u_1^{(0)}, \ldots, u_1^{(j_1-1)}, \ldots, u_m^{(0)}, \ldots, u_m^{(j_m-1)}).$$

The associated vector fields are then:

$$g_0^{(\mathbf{j})} \triangleq g_0^{(\mathbf{j})}(\bar{x}^{(\mathbf{j})}) = \sum_{i=1}^{n} f_i(x, u)\frac{\partial}{\partial x_i} + \sum_{i=1}^{m}\sum_{k=0}^{j_i-1} u_i^{(k+1)}\frac{\partial}{\partial u_i^{(k)}},$$

$$g_i^{(\mathbf{j})} \triangleq g_i^{(\mathbf{j})}(\bar{x}^{(\mathbf{j})}) = \frac{\partial}{\partial u_i^{(j_i)}}, \quad i = 1, \ldots, m$$

so that the system can be written as:

$$\dot{\bar{x}}^{(\mathbf{j})} = g_0^{(\mathbf{j})} + \sum_{i=1}^{m} u_i^{(j_i+1)} g_i^{(\mathbf{j})}$$

We now seek to determine whether the prolonged system is static feedback linearizable. To do so, we first need to define $G_k^{(\mathbf{j})}$ such that $G_k^{(\mathbf{0})}$ equals the definition in Theorem 1, in the following way:

$$G_0^{(\mathbf{j})} \triangleq \{\frac{\partial}{\partial u_1^{(j_1)}}, \ldots, \frac{\partial}{\partial u_m^{(j_m)}}\}, \quad G_{k+1}^{(\mathbf{j})} \triangleq G_k^{(\mathbf{j})} + \mathrm{ad}_{g_0^{(\mathbf{j})}} G_k^{(\mathbf{j})},$$

$\forall k \geq 0$. In order to characterize the evolution of the derived distributions $G_k^{(\mathbf{j})}$, it is essential to distinguish the components that come directly from input prolongations from those produced by successive commutators with the drift vector field. The next proposition, proven in (Lévine, 2024), establishes such a decomposition, which will be fundamental in verifying the pure prolongation flatness (in short P^2-flatness) conditions.

Proposition 2. $G_k^{(\mathbf{j})} = \Gamma_k^{(\mathbf{j})} \oplus \Delta_k^{(\mathbf{j})} \quad \forall k \geq 0$ *where:*

$$\Gamma_k^{(\mathbf{j})} \triangleq \bigoplus_{p=1}^{m} \{ \tfrac{\partial}{\partial u_p^{(j_p-\ell)}} \,|\, \ell = 0, \ldots, \text{máx}(k, j_p - 1\}$$

and

$$\Delta_k^{(\mathbf{j})} \triangleq \sum_{p=1}^{m} \{ ad_{g_0^{(\mathbf{j})}}^{l-j_p} \tfrac{\partial}{\partial u_p} \,|\, l = j_p, \ldots, k\}$$

Given the distributions $G_k^{(\mathbf{j})}$, one could already verify whether the prolonged system is static feedback linearizable by applying Theorem 1. However, the following equivalent result, that appeared in (Lévine, 2024)provides more practical and efficient conditions to check it, avoiding redundant verifications and significantly simplifying its validation process, in particular for computational procedures.

Theorem 2. *A necessary and sufficient condition for Pure-Prolongation-Flatness at* $\bar{0}$ *is that there exists* $\mathbf{j} = (j_1, \ldots, j_m) \in \mathbb{N}^m$, $0 = j_1 \leq \cdots \leq j_m$ *such that:*

1. $\Delta_k^{(\mathbf{j})}$ *is involutive with rank* $\Delta_k^{(\mathbf{j})}$ *locally constant* $\forall k \geq 0$.
2. $[\Gamma_k^{(\mathbf{j})}, \Delta_k^{(\mathbf{j})}] \subset \Delta_k^{(\mathbf{j})} \quad \forall k \geq 0.$
3. $\exists k_\star^{(\mathbf{j})} \leq n + |\mathbf{j}|$ *such that rank* $\Delta_k^{(\mathbf{j})} = n + m \quad \forall k \geq k_\star^{(\mathbf{j})}$.

The idea to determine the minimum prolongation order $\mathbf{j}$ is based on the fact that this order in $\Gamma_k^{(\mathbf{j})}$ is decreasing with k while the one in $\Delta_k^{(\mathbf{j})}$ is increasing, thus requiring to increase $\mathbf{j}$ until item 2 is satisfied for all k.

Note moreover, that the P^2-flatness requires the involutivity of the filtration of subdistributions generated by the non prolonged inputs (item 1).

The following result states the method for constructing the corresponding flat outputs.

Definition 2. *For all* $k \geq 1$, *the* Brunovský controllability indices *are defined as*

$$\kappa_k^{(\mathbf{j})} \triangleq \#\{\ell \mid \rho_\ell^{(\mathbf{j})} \geq k\}, \qquad k = 1, \ldots, m.$$

where:

$$\rho_k^{(\mathbf{j})} := rank\; G_k^{(\mathbf{j})}/G_{k-1}^{(\mathbf{j})}, \qquad \rho_0^{(\mathbf{j})} := rank\; G_0^{(\mathbf{j})} = m.$$

Proposition 3. *The flat outputs* $(y_1, \ldots, y_m)$ *are locally non-trivial solutions of the system of first-order PDEs*

$$\langle G_k^{(\mathbf{j})}, \mathrm{d}y_i \rangle = 0, \qquad k = 0, \ldots, \kappa_i^{(\mathbf{j})} - 2,$$

$$\langle G_{\kappa_i^{(\mathbf{j})}-1}^{(\mathbf{j})}, \mathrm{d}y_i \rangle \neq 0, \qquad i = 1, \ldots, m.$$

And the following mapping is a local diffeomorphism.

$$x^{(\mathbf{j})} \longmapsto \left(y_1, \ldots, y_1^{(k_1^{(\mathbf{j})}-1)}, \ldots, y_m, \ldots, y_m^{(k_m^{(\mathbf{j})}-1)} \right)$$

Based on this result, an algorithm to check whether a system is P^2-flat was outlined in (Lévine, 2024). This algorithm has been refined and programmed in Matlab in (Roca, 2026) and will be the object of a future publication. This computer program allows to check the P^2-flatness of a system in some seconds, reinforcing the idea that achieving differential flatness does not necessarily require the use of highly sophisticated mathematical tools. While dynamic feedback linearization has often been developed within abstract geometric and algebraic frameworks, pure prolongations provide a constructive alternative that just relies on Lie–algebraic computations. This makes the approach conceptually simpler and potentially more accessible for practical use in trajectory planning.

4. Application of the Algorithm to the Simplified Model

As mentioned in the Introduction, we consider the *core system*, a simplified model of an unmanned surface vessel introduced in (de Doná et al., 2018):

$$\begin{cases} \dot{x} = u\cos\psi - v\sin\psi, \\ \dot{y} = u\sin\psi + v\cos\psi, \\ \dot{\psi} = r, \\ \dot{v} = bur - \beta_v v, \end{cases} \tag{10}$$

For this system, the pure prolongation algorithm proceeds as follows:

We first consider prolongations of the form $\mathbf{j} = (j_1, 0)$. For each $k \geq 0$, the distributions $\Delta_k^{(\mathbf{j})}$ and $\Gamma_k^{(\mathbf{j})}$ are computed. For each $k \geq 0$, we determine the minimal value of j_1 for which the conditions of Theorem 2 are satisified, until rank $\Delta_k^{(\mathbf{j})} = n + m = 6$.

For $k = 0$ and $k = 1$, any prolongation with $j_1 \geq 2$ satisfies the conditions in Theorem 2. However, for $k = 2$, there exists no prolongation of the form $\mathbf{j} = (j_1, 0)$ such that $\Delta_2^{(\mathbf{j})}$ is involutive. Consequently, the system is not P^2-flat for prolongations of this form.

We then consider prolongations of the form $\mathbf{j} = (0, j_2)$. In this case, the following results are obtained:

For $k = 0$, any prolongation with $j_2 \geq 1$ fulfills the conditions in Theorem 2. For $k = 1$, the minimal prolongation satisfying the conditions is $j_2 \geq 2$. For $k = 2$, the choice $j_2 = 3$ makes $\Delta_2^{(\mathbf{j})}$ involutive, but the condition $[\Gamma_2^{(\mathbf{j})}, \Delta_2^{(\mathbf{j})}] \subset \Delta_2^{(\mathbf{j})}$ is not met. Both conditions are satisfied for $j_2 \geq 4$. For $k = 3, \ldots, 5$, the same bound $j_2 \geq 4$ works.

The rank $\Delta_k^{(\mathbf{j})}$ is given by rank $\Delta_k^{(\mathbf{j})} = k + 1$.

In summary, any prolongation of the form $\mathbf{j} = (0, j_2)$, with $j_2 \geq 4$ ensures that $\Delta_k^{(\mathbf{j})}$ is involutive for all $k = 0, \ldots, 5$, and that the condition $[\Gamma_k^{(\mathbf{j})}, \Delta_k^{(\mathbf{j})}] \subset \Delta_k^{(\mathbf{j})}$ is also met. Moreover, since rank $\Delta_5^{(\mathbf{j})} = 5 + 1 = 6$, the minimal prolongation for which the prolonged system is static feedback linearizable is $\mathbf{j} = (0, 4)$.

The corresponding flat outputs are obtained by solving the system of partial differential equations described in Proposition 3.

The prolonged state variables are: $[x, y, \psi, v, r, \dot{r}, \ddot{r}, r^{(3)}]$ and the new controls are: $[u, r^{(4)}]$.

The solution of this system leads to $y_1 = \psi$ while y_2 must fulfill the following equations:

$$\frac{\partial y_2}{\partial u} = 0; \quad \frac{\partial y_2}{\partial \ddot{r}} = 0; \quad \frac{\partial y_2}{\partial r^{(3)}} = 0; \quad \frac{\partial y_2}{\partial r^{(4)}} = 0;$$

together with

$$cos(\psi)\frac{\partial y_2}{\partial x} + sin(\psi)\frac{\partial y_2}{\partial y} + br\frac{\partial y_2}{\partial v} = 0;$$

$$-(b-1)rsin(\psi)\frac{\partial y_2}{\partial x} + (b-1)rcos(\psi)\frac{\partial y_2}{\partial y} - b(\dot{r} + \beta_v r)\frac{\partial y_2}{\partial v} = 0;$$

Since the coefficients multiplying the partial derivatives depend only on $(y_1, \dot{y}_1, \ddot{y}_1)$ and not on (x, y, v), we seek a solution of the form:

$$\begin{aligned} y_2 &= A(\psi, r, \dot{r})x + B(\psi, r, \dot{r})y + C(\psi, r, \dot{r})v \\ &= A(y_1, \dot{y}_1, \ddot{y}_1)x + B(y_1, \dot{y}_1, \ddot{y}_1)y + C(y_1, \dot{y}_1, \ddot{y}_1)v \end{aligned}$$

This yields

$$\frac{\partial y_2}{\partial x} = A(\psi, r, \dot{r}), \quad \frac{\partial y_2}{\partial y} = B(\psi, r, \dot{r}), \quad \frac{\partial y_2}{\partial v} = C(\psi, r, \dot{r})$$

and therefore, the relations

$$\begin{cases} cos(\psi)A + sin(\psi)B + brC = 0 & \text{(I)} \\ -(b-1)rsin(\psi)A + (b-1)rcos(\psi)B - b(\dot{r} + \beta_v r)C = 0 & \text{(II)} \end{cases}$$

Choosing $B = -(b-1)rsin(\psi) + (\frac{\dot{r}}{r} + \beta_v)cos(\psi)$ we obtain the flat output

$$\begin{aligned} y_2 = &\left[-(b-1)rcos(\psi) - \left(\frac{\dot{r}}{r} + \beta_v\right)sin(\psi)\right]x \\ &+ \left[-(b-1)rsin(\psi) + \left(\frac{\dot{r}}{r} + \beta_v\right)cos(\psi)\right]y + \left(\frac{b-1}{b}\right)v. \end{aligned}$$

Unlike the first flat output is the heading $y_1 = \psi$, which has a direct geometric interpretation, the second flat output y_2 corresponds to a nontrivial combination of position and velocity variables and does not represent a simple geometric point on the vessel.

5. Flatness of the General Model

No clear result concerning the differential flatness of the most general model of an unmanned surface vessel has been established so far, as suggested in (Degorre et al., 2025). Recall that it reads:

$$\begin{cases} \dot{x} = u\cos\psi - v\sin\psi, \\ \dot{y} = u\sin\psi + v\cos\psi, \\ \dot{\psi} = r, \\ \dot{u} = F_u + avr - \beta_u u, \\ \dot{v} = bur - \beta_v v, \\ \dot{r} = \Gamma_r + cuv - \gamma r. \end{cases} \tag{11}$$

We will prove in this section that it is indeed flat, and that the flat outputs are identical to those of the reduced model studied in the previous section. The reasoning is as follows:

Application of the pure prolongation algorithm to System 11 shows that for $k = 2$, there exists no prolongation of the form $\mathbf{j} = (j_1, 0)$ or $\mathbf{j} = (0, j_2)$ such that $\Delta_2^{(\mathbf{j})}$ is involutive. Consequently, the system is not P^2-flat.

However, if the following feedback law

$$\begin{aligned} v_1 &= F_u + avr - \beta_u u, \\ v_2 &= \Gamma_r + cuv - \gamma r. \end{aligned} \tag{12}$$

is applied to System (11), it becomes:

$$\begin{cases} \dot{x} = u\cos\psi - v\sin\psi, \\ \dot{y} = u\sin\psi + v\cos\psi, \\ \dot{\psi} = r, \\ \dot{u} = v_1, \\ \dot{v} = bur - \beta_v v, \\ \dot{r} = v_2. \end{cases} \tag{13}$$

Note that System (13) is a $(1, 1)$ prolongation of System (8). Therefore, according to Remark 1, since System (8) is Flat by pure prolongation, System (13) is also P^2-flat with a $(0, 4)$ prolongation. The same result can be obtained by directly applying Levine's algorithm to System (13). Finally, the flatness of the latter implies the flatness of (11) thanks to the invertibility of (12).

Application of the Matlab program by (Roca, 2026) leads to the same system of PDEs as in Section 4 and, of course, the flat outputs are exactly the same for both the *core system* and the full model.

Remark 2. *The reasoning applied to System (11) can be extended to the most general case, where the parameters of the system do not have any relationship among them.*

6. Simulations

Since the prolonged system is static feedback linearizable, there exists a diffeomorphism between its state variables

$$(x, y, \psi, u, v, r, v_2^{(0)}, v_2^{(1)}, v_2^{(2)}, v_2^{(3)})$$

and the variables of the flat space

$$(y_1, y_1^{(1)}, y_1^{(2)}, y_1^{(3)}, y_1^{(4)}, y_1^{(5)}, y_2, y_2^{(1)}, y_2^{(2)}, y_2^{(3)})$$

as detailed in Proposition 3.

Our goal is to design a controller such that the system goes from an initial state to a desired final state. To do so, these initial and final conditions are transformed into initial and final conditions of the flat space variables using the diffeomorphism. In this space, the trajectories of $y_1(t)$ and $y_2(t)$ are obtained using polynomial interpolation. Finally, the inputs of the prolonged system are found by inversion of the static feedback law.

Additionally, a closed-loop controller is added to the linear system in order to mitigate model uncertainties and numerical errors.

For simulations, according to (Fossen, 2021), we have considered the following values for the parameters of the system:

$$(a, b, c, \beta_u, \beta_v, \gamma) = (0{,}58, -1{,}72, 0, 10, 15, 10)$$

and initial and final conditions

$$\bar{x}(0) = (0, 0, 0, 0, 0, r_{id}, 0, 0, 0, 0)$$

$$\bar{x}(100) = (10^3, 2 * 10^3, \pi/6, 0, 0, 0, r_{id}, 0, 0, 0, 0)$$

where $\bar{x} = (x, y, \psi, u, v, r, v_2^{(0)}, v_2^{(1)}, v_2^{(2)}, v_2^{(3)})$.

The figures below show that the simulations perform perfectly as expected. The trajectory of the vessel in the XY-plane is shown in Figure 1. Figure 2 depicts the behavior of the inputs Γ_r and F_u.

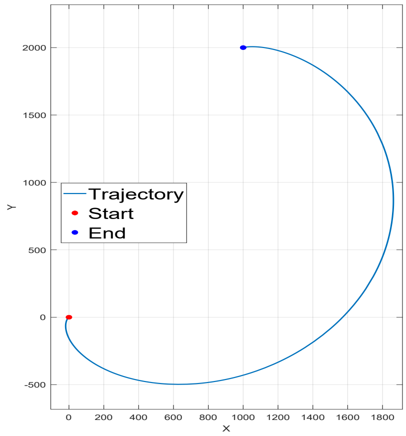

Figura 1: Trajectory in the XY-plane

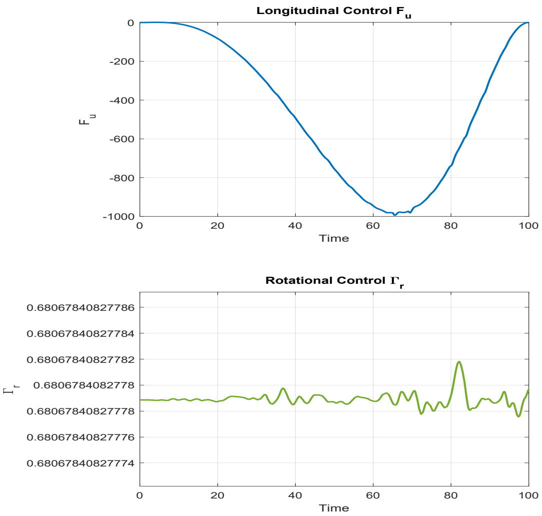

Figura 2: Behavior of the inputs

7. Conclusions

The main contributions of this paper are

- We have proved that the simplified four-degree-of-freedom core model of the USV is flat by pure prolongation. This was achieved by applying the pure prolongation algorithm using a Matlab implementation developed by the authors. The flat outputs have also been obtained by solving the corresponding system of partial differential equations through a particular choice of the solution structure.

- The flatness of the general six-degree-of-freedom unmanned surface vessel has also been demonstrated by proving that this system is obtained from the *core system* by adding an endogenous dynamic compensator, which implies that the flat outputs of both systems are the same.

- The flatness of the general model has also been proven by adding a feedback law, which leads to a system that is a $(1, 1)$ prolongation of the simplified model. Therefore, the general model with the addition of the feedback law is P^2-flat, which is an alternative way of proving the flatness of the general model.

We emphasize that many other systems, previously proven to be differentially flat through complex mathematical derivations, can be shown to be differentially flat by realizing that they can be obtained by adding an endogenous dynamic compensator to simpler systems and then studying the flatness through pure prolongation of these simpler systems. In these cases, the application of a feedback law transforms the systems into pure prolongations of the simplified models, which simplifies the study of their flatness. A detailed study of these examples will appear in a future publication.

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